One-year premium risk and emergence pattern of ultimate loss based on conditional distribution

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About the speaker

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Introduction

Let $(X_1, X_2, \ldots, X_n)$ denote the **cumulative payments associated with a given accident year**, where X_i denotes the claims paid up to the i-th development year. The random variables $(X_1, \ldots, X_n)$ are dependent.

At each point of time $k = 1, \ldots, n-1$, the insurer predicts the value of the aggregate claims by calculating $\mathbb{E}[X_n | X_1, \ldots, X_k]$. The expected value

$$BE_k = \mathbb{E}[X_n | X_1, \ldots, X_k]$$

is called the **best estimate of the ultimate loss**.

Introduction

Insurance companies are exposed to **premium** and **reserve risk**:

- **Premium risk** - related to the losses resulting from the premiums that are to be earned in the following year.
- **Reserve risk** - related to the adequacy of the current volumes of the claims reserves.

We also differ between the notion of **ultimate** and **one-year risk**. For premium risk we understand them as the risk that the premiums earned in a given year are not sufficient to cover:

- For **ultimate risk** the losses paid in an infinite time horizon (the so-called ultimate loss) - described by X_n.
- For **one-year risk** the losses paid in the first year and the reserve set at the end of the first year - described by BE_1.

Introduction

For classic reserve risk models, where we know the distribution of X_1 and $X_{i+1}|X_i$, it is clear how to perform a **forward simulation** of $(X_1, \ldots, X_n)$ and we know the relation between X_n and X_1. Such a forward simulation scheme is investigated e.g. in [Wüthrich, Merz(2015)] where the relations between **one-year reserve risk** and the ultimate reserve risk is discussed in details.

The new problem which we study in this paper is how to model the **one-year premium risk** and the ultimate premium risk (BE_1, X_n) by generating them in a **backward simulation** starting with the ultimate loss X_n for the new accident year.

In reserve risk models, so called **one-year claims development results** $CDRs$ are investigated. We want to use a counterpart for the one-year premium risk, which is the **technical result for the new accident year** defined as the difference between premiums and claims. Since the premiums include an expected profit margin, we replace them with $E[X_n]$ in our definition, having as a result $X_n - E[X_n]$ as the modelled variable for ultimate risk and $BE_1 - E[BE_1]$ for one-year risk.

Introduction

Motivation:

1. One-year risk needs to be investigated by the companies for **Solvency II risk capital.** Many companies have already created models for simulating their ultimate losses and we can modify them into **one-year models** by means of backward simulation.
2. From business point of view, the unconditional distribution of X_n is well-understood by decision makers, is used in all planning reports, is the basis of pricing, long-term risk analysis and allows for plausibility checks of the results.

We will model the relations between one-year and ultimate premium risk by finding the **true emergence pattern**, which is defined as the conditional distribution $BE_1|X_n$.

From the conditional distribution of $BE_1|X_n$, we next derive the unconditional distribution of BE_1 used for quantifying the *true* (unconditional) one-year premium risk. This will allows us to study the **true relation between the one-year and ultimate premium risk** in models with various distributions of the ultimate loss and various claims development processes.

Emergence pattern

We follow the approach of [England(2012)] and [Bird, Cairns(2011)], who introduce the concept of an emergence pattern of the ultimate loss. They postulate the following relation of BE_1 and X_n by using a simple linear function:

$$BE_1^{ep} = \alpha X_n + (1-\alpha)\mathbb{E}[X_n],$$

where α is called **an emergence factor** and $\alpha \in (0,1)$.

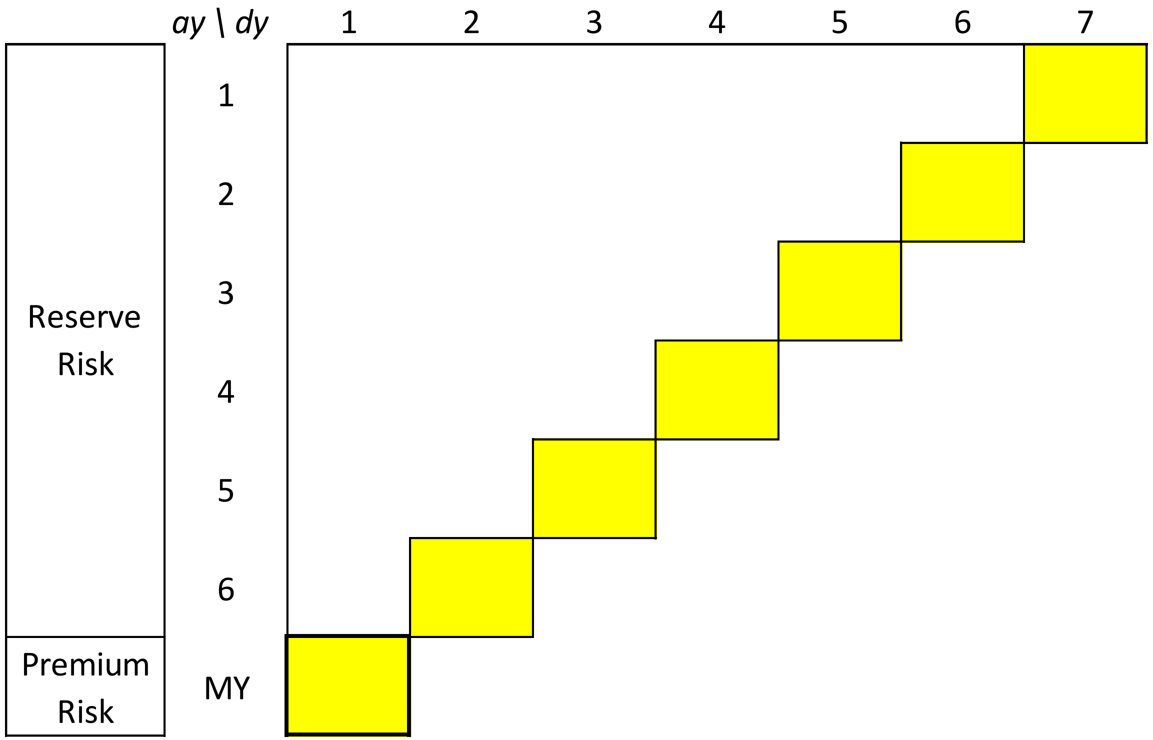
To calculate the α parameter we follow the idea of [England(2012)], [Bird, Cairns(2011)]

$$\alpha = \frac{SD[BE_1]}{SD[X_n]} = \frac{SD\big[BE_1 - E[BE_1]\big]}{SD\big[X_n - E[X_n]\big]}$$

The estimation of standard deviation of BE_1 and X_n is done in two steps:

- We estimate the distribution of development factors $(X_{i+1}|X_i)_{i=1}^{n-1}$ or $(X_{i+1} \; - X_i)_{i=1}^{n-1}$ from the historical losses in a run-off triangle in a claims reserving model.
- We estimate the unconditional distribution of X_1 using e.g. an additive model, which is usually related to the planned volume of exposure, coming from the financial plans.

Emergence pattern



Additional assumption:

- We do not consider estimation error and we assume that all parameters of the claims development process are given - as a result one-year premium risk can be investigated independently of one-year reserve risk.

Key goal:

- Analyse probabilistic properties of the one-year premium risk and the ultimate premium risk implied by various claims development processes.

Emergence pattern

$$BE_1^{ep} = \alpha X_n + (1-\alpha)\mathbb{E}[X_n],$$

Theorem 1. *We have the following properties of the emergence pattern:*

1. $\mathbb{E}[BE_1^{ep}] = \mathbb{E}[X_n]$ and $Var[BE_1^{ep}] = \alpha^2 Var[X_n] < Var[X_n]$,
2. $VaR_\gamma[BE_1^{ep} - \mathbb{E}[BE_1^{ep}]] = \alpha VaR_\gamma[X_n - \mathbb{E}[X_n]] < VaR_\gamma[X_n - \mathbb{E}[X_n]]$,
3. If X_n has a light-tailed (subexponential with all moments finite) distribution, then BE_1^{ep} has a light-tailed (subexponential with all moments finite) distribution,
4. If X_n has a heavy-tailed distribution with tail index θ, then BE_1^{ep} has a heavy-tailed distribution with tail index θ, and we have the limit $\lim_{\gamma\to 1} \frac{VaR_\gamma[BE_1^{ep} - \mathbb{E}[BE_1^{ep}]]}{VaR_\gamma[X_n - \mathbb{E}[X_n]]} = \alpha$.

Emergence pattern

Theorem 2.

1. The one-year risk is lower than the ultimate risk at all confidence levels.
2. The one-year risk decreases linearly in α when the emergence factor α decreases at all confidence levels.
3. The distributions of the one-year risk and the ultimate risk have the same tail behaviour.

Disadvantages of the emergence pattern approach:

1. The emergence pattern is true only if BE_1 is perfectly linearly correlated with X_n, i.e. if $\rho(BE_1, X_n) = 1$.
2. The conditional distribution of $BE_1^{ep}|X_n = x$ is degenerate.
3. The true relation between $VaR_\gamma[BE_1]$ and $VaR_\gamma[X_n]$ varies across the models and may not be linear in α. Additionally, it may not be equal to the relation of the $SD[BE_1]$ and $SD[X_n]$.

Incremental Loss Ratio Gaussian Model

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Firstly, we consider **Incremental Loss Ratio model with Gaussian incremental losses**.

$$X_j = \sum_{i=1}^{j} \epsilon_i, \qquad \text{where: } \epsilon_i \sim N\left(Em_i, E\sigma_i^2\right) \ \text{ for } i \in \{1, \dots, n\},$$
$$\text{and: } \epsilon_i \perp\!\!\!\perp \epsilon_j \quad \text{for } \ i \neq j \in \{1, \dots, n\}.$$

E denotes the exposure in the accident year under consideration, and ϵ_i represents the incremental loss in development year i. We will denote

$$m = \sum_{i=1}^{n} m_i, \qquad \sigma^2 = \sum_{i=1}^{n} \sigma_i^2.$$

In this reserve risk model the best estimate of the ultimate loss after the first year is calculated by

$$BE_1 = \mathbb{E}[X_n | X_1] = X_1 + E(m - m_1).$$

Incremental Loss Ratio Gaussian Model

Proposition 1. *Let us consider the Incremental Loss Ratio Gaussian model of claims development. We have the following loss distributions:*

- $X_n \sim N(Em, E\sigma^2)$,
- $BE_1 \sim N(Em, E\sigma_1^2)$,
- $X_1|X_n = x \sim N(\frac{\sigma_1^2}{\sigma^2}(x - E(m - m_1)) + \frac{\sigma^2 - \sigma_1^2}{\sigma^2} Em_1;\ E\frac{\sigma_1^2(\sigma^2 - \sigma_1^2)}{\sigma^2})$,
- $BE_1|X_n = x \sim N(\frac{\sigma_1^2}{\sigma^2} x + \frac{\sigma^2 - \sigma_1^2}{\sigma^2} Em; E\frac{\sigma_1^2(\sigma^2 - \sigma_1^2)}{\sigma^2})$.

Incremental Loss Ratio Gaussian Model

We are able to **improve the emergence pattern formula** so that it yields the correct conditional distribution of $BE_1|X_n$ and unconditional distribution of BE_1 in the reserve risk model and does not depend explicitly on the distributions of X_1 and $X_{i+1}|X_i$ - it depends only on the distribution of X_n and the emergence factor α.

Theorem 3. *Let us set*

$$\mu_{X_n} = \mathbb{E}[X_n], \qquad \sigma_{X_n}^2 = Var[X_n], \qquad \alpha = \frac{SD[BE_1]}{SD[X_n]}.$$

We consider the ILR gaussian model with $X_n \sim N(\mu_{X_n}, \sigma_{X_n}^2)$. We have the following distributions of the best estimate of the ultimate loss:

- $BE_1|X_n = x \sim N(\alpha^2 x + (1-\alpha^2)\mu_{X_n}; \alpha^2(1-\alpha^2)\sigma_{X_n}^2)$,
- $BE_1 \sim N(\mu_{X_n}, \alpha^2\sigma_{X_n}^2)$.

Incremental Loss Ratio Gaussian Model

Following the emergence pattern formula we have

$$BE_1^{ep} = \alpha X_n + (1-\alpha)\mu_{X_n} \quad \text{and} \quad BE_1^{ep} \sim N(\mu_{X_n}, \alpha^2 \sigma_{X_n}^2).$$

Theorem 4. *Let us consider the ILR Gaussian model.*

1. The emergence pattern formula yields the proper distribution of the one-year risk,
2. The one-year risk is lower than the ultimate risk at all confidence levels,
3. The one-year risk decreases linearly in α when the emergence factor α decreases at all confidence levels,
4. The distributions of the one-year risk and the ultimate risk have the same tail behaviour.

Multiplicative lognormal model

Secondly, we consider a **multiplicative loss model** where the development factors are modelled with **lognormal distributions**. We deal with the cumulative payments determined by the reserve risk model:

$$X_j = X_{j-1} \cdot \epsilon_j, \text{where:} \, \epsilon_i \sim LogN(m_i, \sigma_i^2) \quad \text{for} \, i \in \{1, \ldots, n\},$$
$$X_1 = \epsilon_1 \text{and:} \, \epsilon_i \perp\!\!\!\perp \epsilon_j \quad \text{for} \, i \neq j \in \{1, \ldots, n\}.$$

We will denote

$$m = \sum_{i=1}^{n} m_i, \qquad \sigma^2 = \sum_{i=1}^{n} \sigma_i^2.$$

In this reserve risk model the best estimate of the ultimate loss after the first year is calculated as

$$BE_1 = \mathbb{E}[X_n | X_1] = X_1 e^{m - m_1 + \frac{1}{2}(\sigma^2 - \sigma_1^2)}.$$

Multiplicative lognormal model

Theorem 5. *For* $X_n \sim LogN$ *with the expected value* μ_{X_n} *and variance* $\psi_{X_n}^2 \mu_{X_n}^2$

$$\mu_{X_n} = \mathbb{E}[X_n], \qquad \psi_{X_n} = \frac{SD[X_n]}{\mathbb{E}[X_n]}, \qquad \alpha = \frac{SD[BE_1]}{SD[X_n]}.$$

We receive the distributions of the best estimate of the ultimate loss:

$$BE_1 | X_n = x \sim LogN\left(\tilde{\alpha}^2 \log(x) + (1-\tilde{\alpha}^2)\left(\tilde{m} + \frac{\tilde{\sigma}^2}{2}\right); \tilde{\alpha}^2(1-\tilde{\alpha}^2)\tilde{\sigma}^2\right),$$

$$BE_1 \sim LogN\left(\tilde{m} + (1-\tilde{\alpha}^2)\frac{\tilde{\sigma}^2}{2}; \tilde{\alpha}^2\tilde{\sigma}^2\right),$$

where the parameters are

$$\tilde{m} = log(\mu_{X_n}) - \frac{1}{2}\log(1+\psi_{X_n}^2), \qquad \tilde{\sigma}^2 = \log(1+\psi_{X_n}^2),$$

$$\tilde{\alpha}^2 = \frac{\log(1+\alpha^2\psi_{X_n}^2)}{\log(1+\psi_{X_n}^2)}.$$

Multiplicative lognormal model

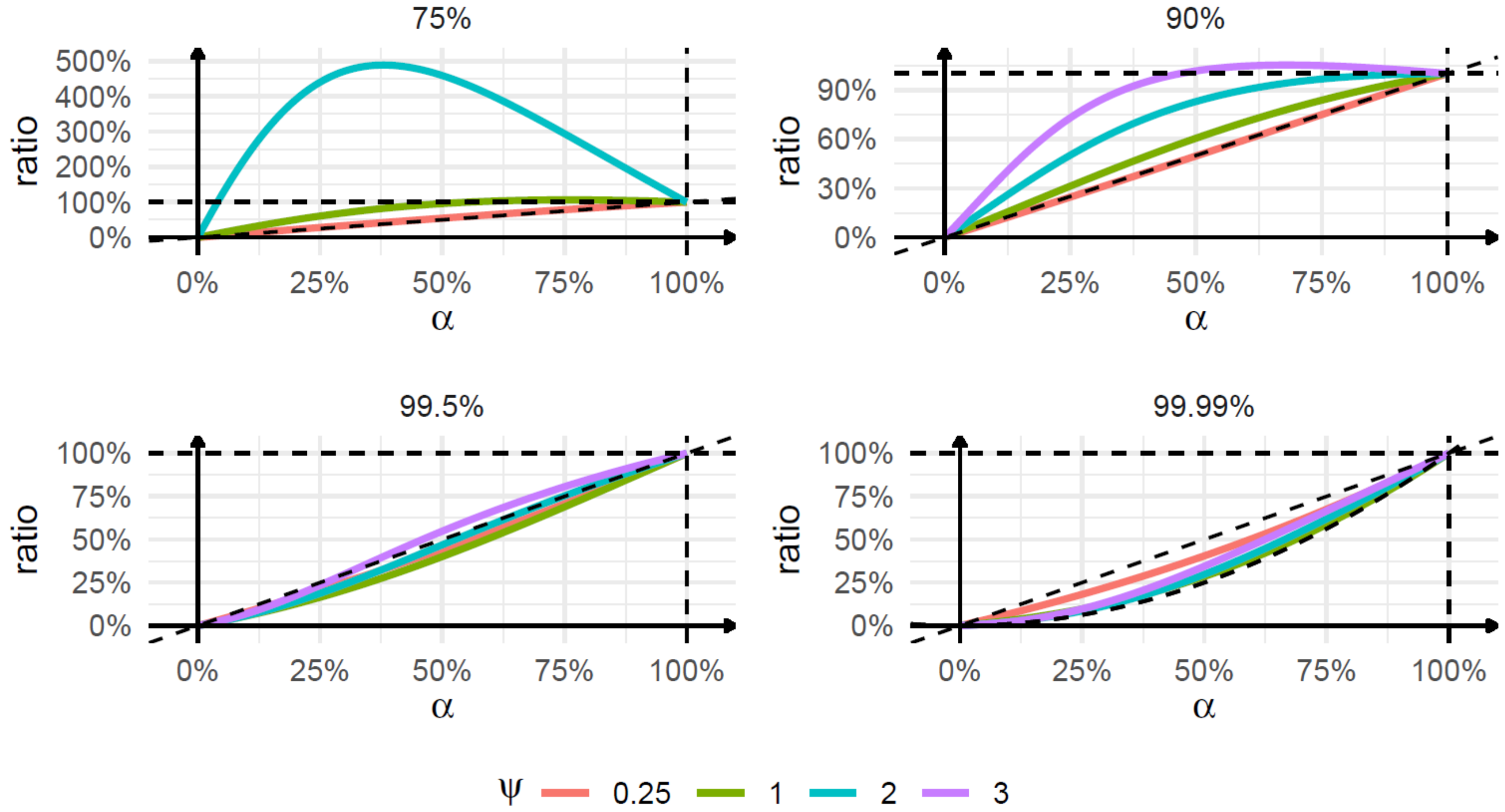
Following the emergence pattern formula we have

$$BE_1^{ep} \sim \alpha \cdot LogN(\tilde{m}, \tilde{\sigma}^2) + (1-\alpha)\mu_{X_n}.$$

Theorem 6. *Let us consider the multiplicative lognormal model.*

1. The emergence pattern formula underestimates the true one-year risk at low confidence levels and overestimates the true one-year risk at high confidence levels,
2. The true one-year risk is lower than the ultimate risk at high confidence levels but the true one-year risk is higher than the ultimate risk at low confidence levels,
3. The true one-year risk vanishes compared to the ultimate risk for the confidence level in the limit $\gamma \to 1$, i.e., $\lim_{\gamma\to 1} \frac{VaR[BE_1 - \mathbb{E}[BE_1]]}{VaR[X_n - \mathbb{E}[X_n]]} = 0$. In practical examples, the true one-year risk at very high confidence levels can decreases as fast as at order α^2 when the emergence factor α decreases.

Multiplicative lognormal model



The ratios $\frac{VaR_\gamma[BE_1-\mathbb{E}[BE_1]]}{VaR_\gamma[X_n-\mathbb{E}[X_n]]}$ in the multiplicative lognormal model.

Emergence pattern – arbitrary X_n

It may be difficult to specify a priori the joint multivariate distribution for cumulative payments $(X_1, X_2, \dots, X_n)$, which lead to a pre-specified distribution of the ultimate loss X_n. That is why, our next step is to modify the reparametrized approach, so that we may use an arbitrary distribution of the ultimate loss.

What we suggest is to **keep the conditional distribution of $BE_1|X_n$ and use any unconditional distribution of the ultimate loss X_n.**

Firstly, we have a flexible and interpretable probabilistic **model, where we can switch from the ultimate risk to the one-year risk** and which can be used in Solvency II premium risk modelling.

Secondly, we can **investigate properties of the one-year risk vs. the ultimate risk** in various claims development models, beyond the models we know from the claims reserving literature.

Emergence pattern – arbitrary X_n

For the ILR Gaussian model we have the following representation:

$$BE_1 = \alpha^2 X_n + (1-\alpha^2)\mu_{X_n} + \sqrt{\alpha^2(1-\alpha^2)}\sigma_{X_n}\xi,$$

where $\xi \sim N(0,1), X_n \sim N(\mu_{X_n}, \sigma_{X_n}^2)$, and ξ is independent of X_n.

It can be seen as an extension of the classical emergence pattern formula, where we simply add a Gaussian noise in order to have a non-degenerate distribution of $BE_1|X_n = x$.

For the multiplicative lognormal model we have the following representation:

$$BE_1 = (X_n)^{\tilde{\alpha}^2} e^{(1-\tilde{\alpha}^2)(\tilde{m}+\frac{\tilde{\sigma}^2}{2})} e^{\sqrt{\tilde{\alpha}^2(1-\tilde{\alpha}^2)}\tilde{\sigma}\xi},$$

where $\xi \sim N(0,1), X_n \sim LogN(m, \sigma^2)$, and ξ is independent of X_n.

It can be seen as an extension of the classical emergence pattern formula, where we allocate the ultimate loss X_n to BE_1 with a random scaling factor in order to have a non-degenerate distribution of $BE_1|X_n = x$.

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Thank you for your attention

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