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Modeling Multi-country Mortality Dependence by a Vine Copula

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About the speaker

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- Japanese subsidiary of Prudential Financial, Inc.
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Introduction

- Multi-country mortality dependence attracts the attention of insurers operating life insurance or annuity business in multiple countries.
- When implementing a sophisticated enterprise risk management (ERM) program, it is crucial to model the structure of such dependence accurately.
- Elliptic and Archimedean copulas are often used for risk aggregation in advanced ERM.
- However, these well-known copulas cannot always flexibly capture complex tail dependence, especially under certain stressed situations.
- This study proposes modeling multi-country mortality dependence by a vine copula, which provides greater flexibility and efficiently characterizes the dependence structure.

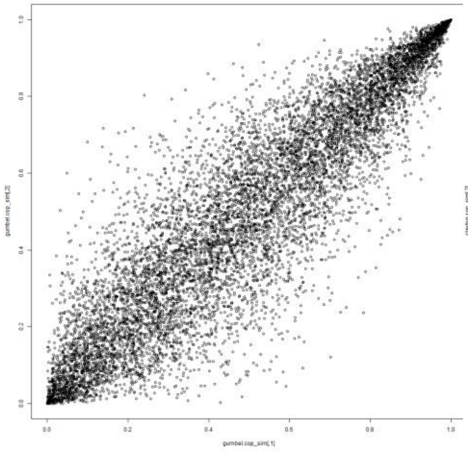
What is a copula ?

- A copula is the joint distribution of distribution functions of random valuables.

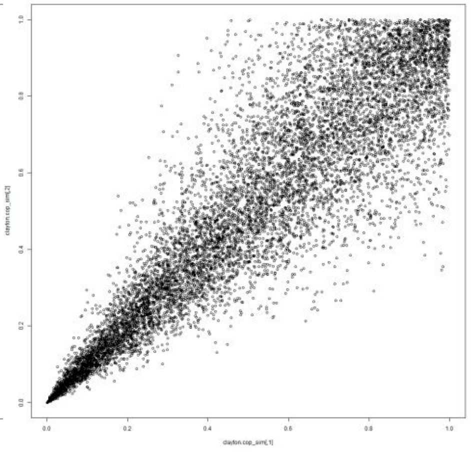
$$F(x_1, x_2) = C\big(F_1(x_1), F_2(x_2)\big).$$

- A copula is able to flexibly express the dependence structure between bivariate random valuables, especially the tail dependence.
- Follows are the plots of well-known bivariate copulas(Kendall’s tau =0.75).

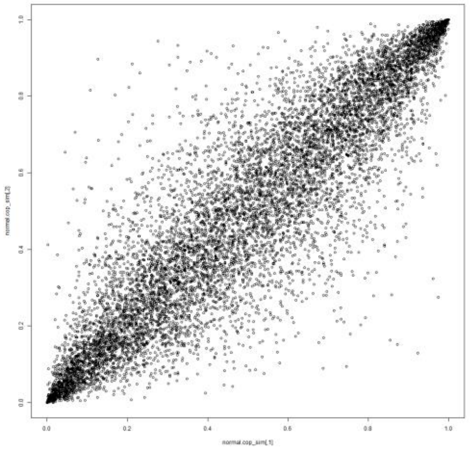
Gumbel copula



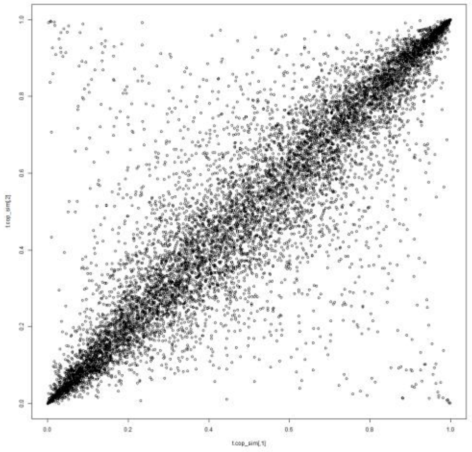
Clayton copula



Gaussian copula



T copula



Limitation of well-known copulas

- Archimedean copulas(such as Gumbel copula and Clayton copula) have only one parameter, so these copulas cannot express the different dependence between each pair of random variables.
- Gaussian copula does not have tail dependence and underestimates the risk.
- T copula is better, but not flexible because t copula expresses the tail dependence by only one parameter (the degree of freedom).

Copulas	# parameters	Lower tail dependence	Upper tail dependence
Gumbel copula	1	0	$2-2^{1/\alpha}$
Clayton copula	1	$2^{-1/\alpha}$	0
Gaussian copula	n(n-1)/2	0	0
T copula	n(n-1)/2+1	$2t_{v+1}\left(-\sqrt{\frac{(v+1)(1-\rho)}{1+\rho}}\right)$	$2t_{v+1}\left(-\sqrt{\frac{(v+1)(1-\rho)}{1+\rho}}\right)$

n : the number of variables v : the degree of freedom ρ : correlation coefficient

What is a vine copula ?

In applying Sklar's theorem proposed by Sklar(1959), one can construct the joint probability density function of random variables X_1 and X_2 as follows:

$$f(x_1, x_2) = c_{1,2}\big(F_1(x_1), F_2(x_2)\big) f_1(x_1) f_2(x_2). \qquad (1)$$

We can also express the joint probability density function using conditional probability as follows:

$$f(x_1, x_2) = f_{1|2}(x_1|x_2) f_2(x_2). \qquad (2)$$

Combining equation (1) and (2), $f_{1|2}(x_1|x_2)$ is shown as follows:

$$f_{1|2}(x_1|x_2) = c_{1,2}\big(F_1(x_1), F_2(x_2)\big) f_1(x_1). \qquad (3)$$

What is a vine copula ?

We can express the joint probability density function of random valuables (X_1, X_2 and , X_3) using the conditional probability density function as follows:

$$f(x_1, x_2, x_3) = f_3(x_3) \boldsymbol{f_{2|3}(x_2|x_3)} \boldsymbol{f_{1|2,3}(x_1|x_2, x_3)}. \qquad (1)$$

Following the last page, $\boldsymbol{f_{2|3}(x_2|x_3)}$ can be written as follows:

$$\boldsymbol{f_{2|3}(x_2|x_3)} = c_{2,3}\left(F_2(x_2), F_3(x_3)\right) f_2(x_2). \qquad (2)$$

$\boldsymbol{f_{1|2,3}(x_1|x_2, x_3)}$ can be obtained by a conditional bivariate copula density function

$$\begin{aligned} \boldsymbol{f_{1|2,3}(x_1|x_2, x_3)} &= c_{1,3|2}\left(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)\right) \boldsymbol{f_1(x_1|x_2)} \\ &= c_{1,3|2}\left(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)\right) c_{1,2}\left(F_1(x_1), F_2(x_2)\right) f_1(x_1). \end{aligned} \qquad (3)$$

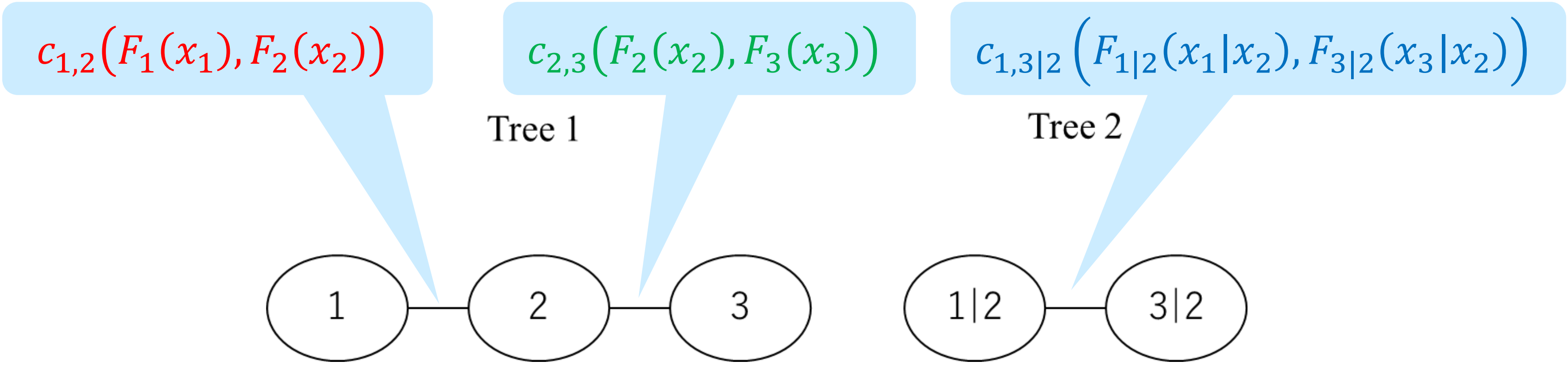
Combining (1),(2), and (3), we obtain

$$\begin{aligned} f(x_1, x_2, x_3) = {} & c_{1,3|2}\left(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)\right) c_{1,2}\left(F_1(x_1), F_2(x_2)\right) \\ & \times c_{2,3}\left(F_2(x_2), F_3(x_3)\right) f_1(x_1) f_2(x_2) f_3(x_3). \end{aligned} \qquad (4)$$

This feature can be easily generalized to n dimensions.

What is a vine copula ?

- Such dependence structure can be graphically expressed by vines, the concept of graph theory.
- A vine copula expresses the node as a random variable and the edge as a conditional bivariate copula.
- The application for market risk and credit risk modeling has been studied in recent years.



$$f(x_1, x_2, x_3) = c_{1,3|2}\left(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)\right) c_{1,2}\left(F_1(x_1), F_2(x_2)\right) \times c_{2,3}\left(F_2(x_2), F_3(x_3)\right) f_1(x_1) f_2(x_2) f_3(x_3)$$

The overview of this research

- Express the mortality rate of 13 countries(12 European countries and Australia) by a Lee-Carter model.
- Model the joint distribution of mortality improvement parameters of each country by a vine copula.
- Compare our model to the models by benchmark copulas.

Estimation of mortality rates (Lee-Carter model) → Estimation of improvement parameters (time series model) → Estimation of the joint distribution （vine copula）

Estimation of mortality rates

Estimation of mortality rates
(Lee-Carter model)

→

Estimation of improvement parameters
(time series model)

→

Estimation of the joint distribution
（vine copula）

Data & Methodology

Data source	:	Human Mortality Database
Countries	:	12 European countries ＋Australia(AUS)
Age	:	From 65 to 90 inclusive
Sex	:	Male
Year	:	1921 ～ 2014

- We model the marginal dynamic of the mortality rates for each country based on a Lee-Carter model.
- In the modeling of this study, we use R package ”StMoMo” and “VineCopula”.

Lee-Carter model

Mortality sensitivity by κ_t

$$log(q_{x,t}) = \alpha_x + \beta_x \cdot \kappa_t + e_{x,t}$$

Logarithm of Base q_x

Improvement parameter by year

$q_{x,t}$: base mortality rates at age x and time t

α_x : average profile of mortality

κ_t: mortality changes over time

β_x: how much each age group mortality changes when κ_t^j changes

$e_{x,t}$: error term which reflects the effects not captured by the model

Estimation of morality rates of 13 countries

$$log(q_{x,t}^1) = \alpha_x^1 + \beta_x^1 \kappa_t^1 + e_{x,t}^1$$ Australia

$$log(q_{x,t}^2) = \alpha_x^2 + \beta_x^2 \kappa_t^2 + e_{x,t}^2$$ Belgium

$$log(q_{x,t}^3) = \alpha_x^3 + \beta_x^3 \kappa_t^3 + e_{x,t}^3$$ Nederland

$$log(q_{x,t}^4) = \alpha_x^4 + \beta_x^4 \kappa_t^4 + e_{x,t}^4$$ England & Wales

$$log(q_{x,t}^5) = \alpha_x^5 + \beta_x^5 \kappa_t^5 + e_{x,t}^5$$ Denmark

$$\vdots$$ ⋮

$$log(q_{x,t}^{13}) = \alpha_x^{13} + \beta_x^{13} \kappa_t^{13} + e_{x,t}^{13}$$ Iceland

We fit the Lee-carter model to the mortality data of each country.

Estimation of improvement parameters

Estimation of mortality rates
(Lee-Carter model)

→

Estimation of improvement parameters
(time series model)

→

Estimation of the joint distribution
（vine copula）

Estimation of future mortality rates

$$\log(q_{x,t}^{1}) = \alpha_x^1 + \beta_x^1 \kappa_t^1 + e_{x,t}^1$$ Australia

$$\log(q_{x,t}^{2}) = \alpha_x^2 + \beta_x^2 \kappa_t^2 + e_{x,t}^2$$ Belgium

$$\log(q_{x,t}^{3}) = \alpha_x^3 + \beta_x^3 \kappa_t^3 + e_{x,t}^3$$ Nederland

$$\log(q_{x,t}^{4}) = \alpha_x^4 + \beta_x^4 \kappa_t^4 + e_{x,t}^4$$ England & Wales

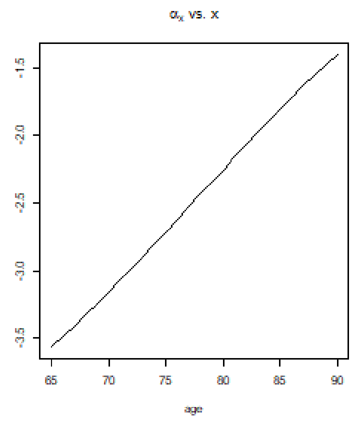
$$\log(q_{x,t}^{5}) = \alpha_x^5 + \beta_x^5 \kappa_t^5 + e_{x,t}^5$$ Denmark

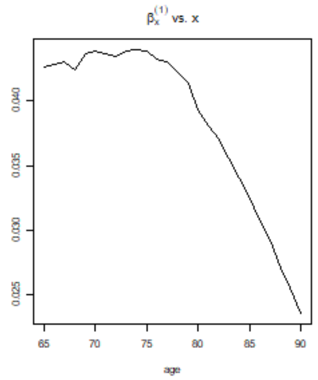
$$\vdots$$ $\vdots$

$$\log(q_{x,t}^{13}) = \alpha_x^{13} + \beta_x^{13} \kappa_t^{13} + e_{x,t}^{13}$$ Iceland

We assume α_x^j and β_x^j are constant over the future period, and model κ_t^j as a time series model.

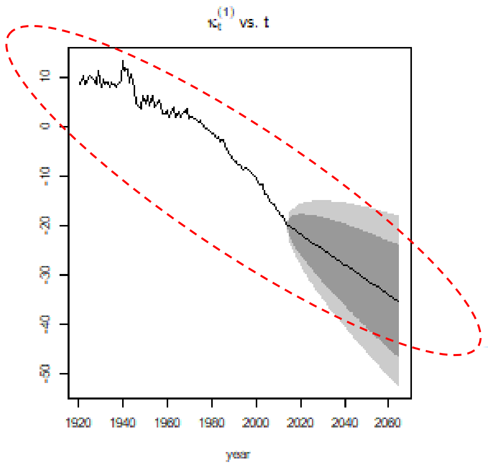
The result of estimation (ex. France)





α_x : Logarithm of base q_x

β_x : Mortality sensitivity by κ_t



$$log(q_{x,t}) = \alpha_x + \beta_x \cdot \kappa_t + e_{x,t}$$

The movement of κ_t looks straight
⇒Random walk with a drift term

$$\kappa_t = \kappa_{t-1} + \mu + \varepsilon_t$$

κ_t : Improvement parameter by year

Modeling the movement of the improvement parameter

$$\kappa_t^1 = \kappa_{t-1}^1 + \mu^1 + \varepsilon_t^1,\quad \varepsilon_t^1 \sim N(0, \sigma_1^2)$$

$$\kappa_t^2 = \kappa_{t-1}^2 + \mu^2 + \varepsilon_t^2,\quad \varepsilon_t^2 \sim N(0, \sigma_2^2)$$

$$\kappa_t^3 = \kappa_{t-1}^3 + \mu^3 + \varepsilon_t^3,\quad \varepsilon_t^3 \sim N(0, \sigma_3^2)$$

$$\kappa_t^4 = \kappa_{t-1}^4 + \mu^4 + \varepsilon_t^4,\quad \varepsilon_t^4 \sim N(0, \sigma_4^2)$$

$$\kappa_t^5 = \kappa_{t-1}^5 + \mu^5 + \varepsilon_t^5,\quad \varepsilon_t^5 \sim N\left(0, \sigma_5^2\right)$$

$$\vdots$$

$$\kappa_t^{13} = \kappa_{t-1}^{13} + \mu^{13} + \varepsilon_t^{13},\quad \varepsilon_t^{13} \sim N(0, \sigma_{13}^2)$$

Each error term follows a normal distribution.

In this study, we model the joint distribution of ε_t^i by a vine copula.

Estimation of the joint distribution

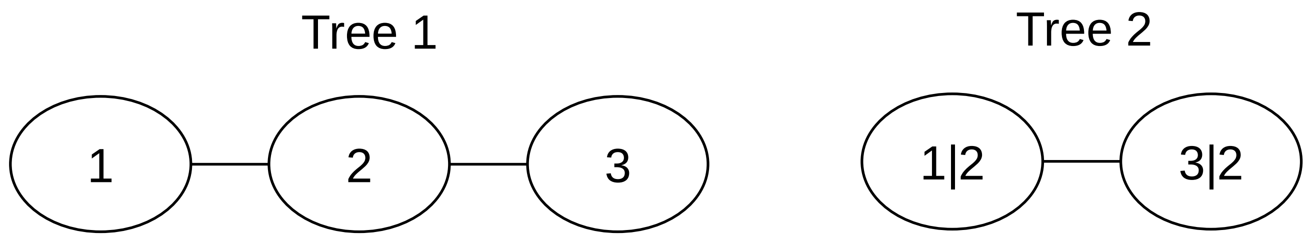
Estimation of mortality rates
(Lee-Carter model)

Estimation of improvement parameters
(time series model)

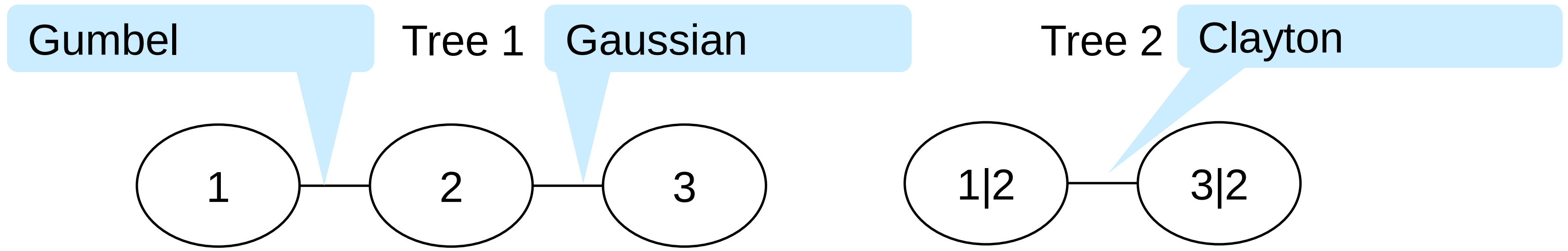
Estimation of the joint distribution
（vine copula）

The process of the modeling by a vine copula

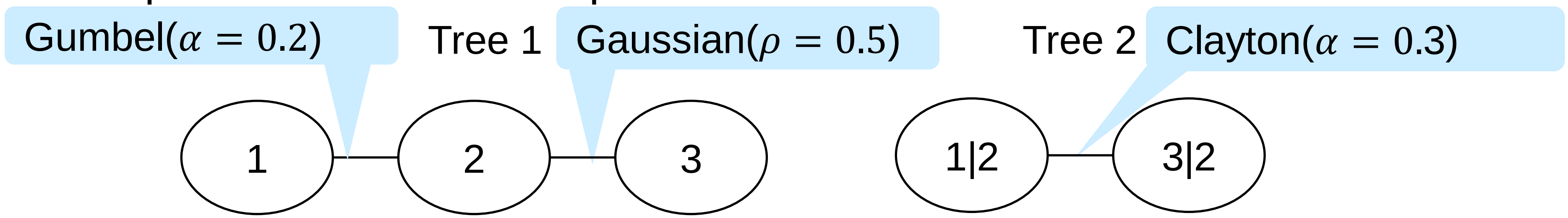
1. Determine the tree structure.



2. Determine the copula families applied to each edge of the vine.



3. Estimate the parameters of each copula.



Modeling by a vine copula

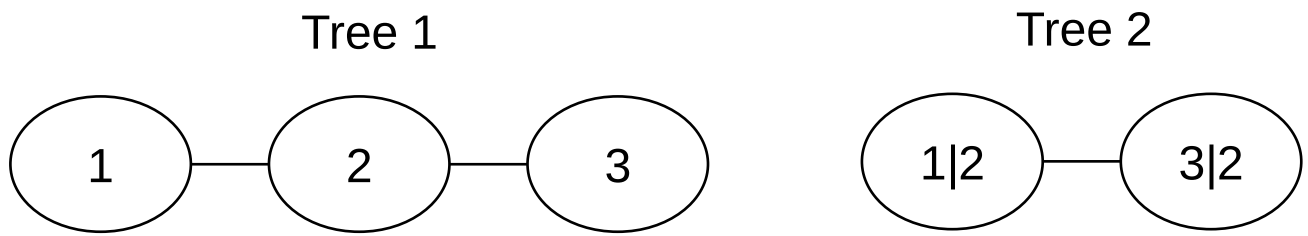
- Although a vine copula has flexibility, the number of tree structures is too huge to choose.
- The following table illustrates the super-exponential growth of the number of structures for dimensions $n = 2, \dots, 10$.
- An automatic algorithm is necessary for modeling by a vine copula.

Dimension (n)	# tree structures	Tree × copula*
2	1	7
3	3	1,029
4	24	2,823,576
5	480	1.3559 e+11
6	23,040	1.0938 e+17
7	2,580,480	1.4413 e+24
8	660,602,880	3.0387 e+32
9	3.8051e+11	1.0090 e+42
10	4.8705e+14	5.2118 e+52

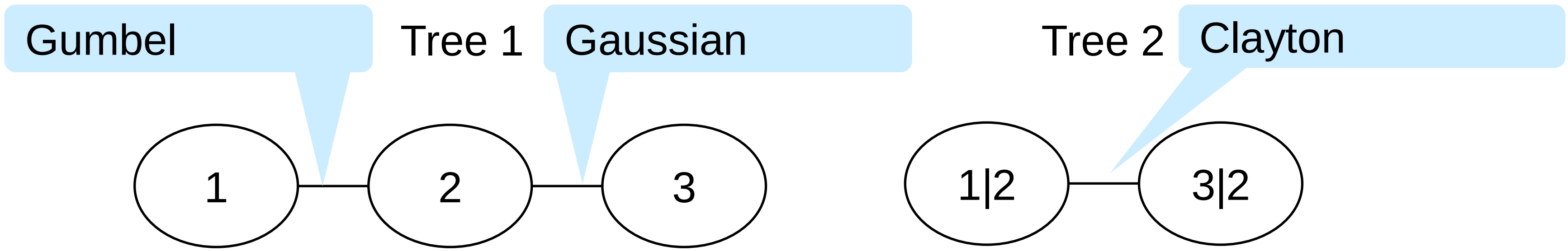
*When there are 7 options for copula.

Sequential method proposed by Dißmann et al. (2013)

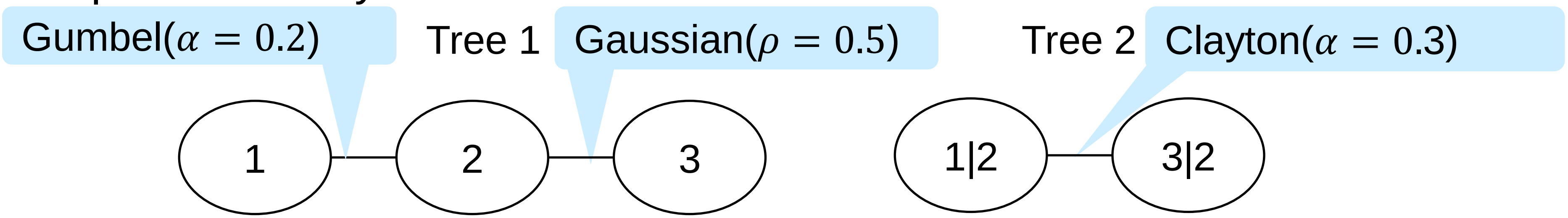
1. Choose the tree structure that maximizes the absolute value of each pair's Kendall's tau.



2. Select the bivariate copula for each edge of the tree based on AIC.



3. Estimate the parameters by maximum likelihood estimation.



Result and consideration

Dependence structure by the vine copula

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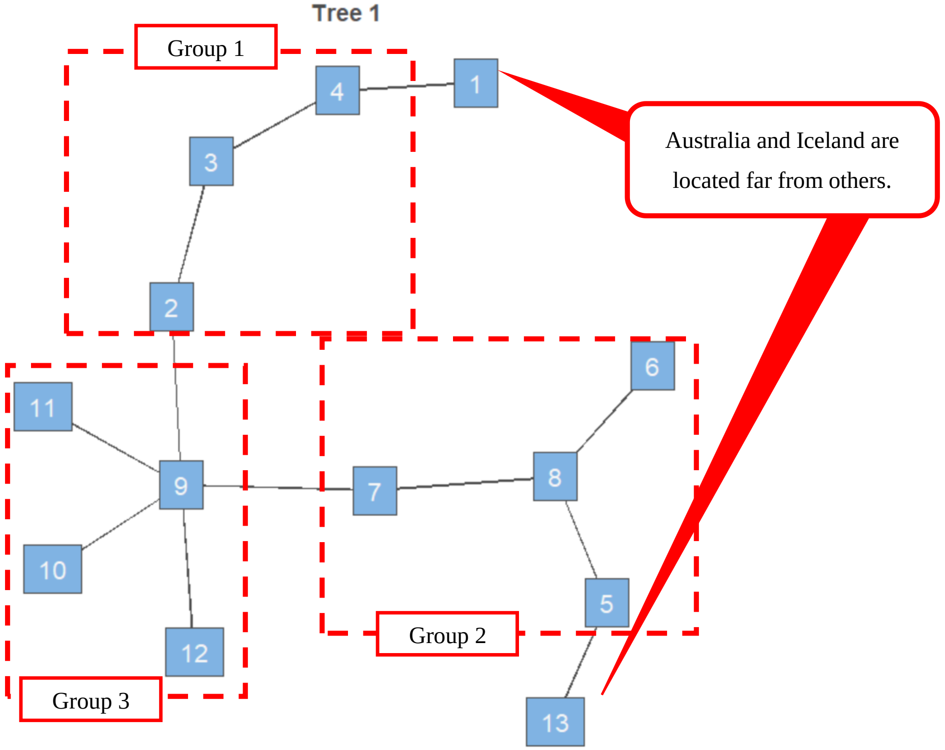
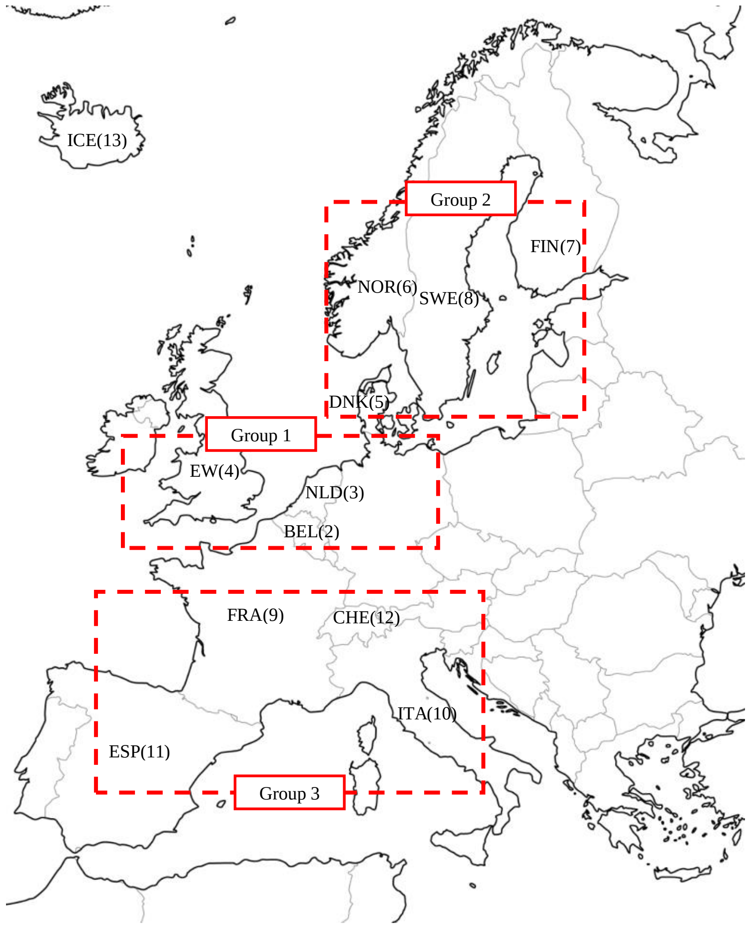


1 <-> AUS, 2 < -> BEL, 3 <-> NLD, 4 < -> EW, 5 < -> DNK, 6 < -> NOR, 7 < -> FIN,
8 <-> SWE, 9 < -> FRA, 10 < -> ITA, 11 < -> ESP, 12 <-> CHE, 13 < -> ICE



By using the sequential method, we obtain a dependence structure resembling the actual geographical relationships, which is intuitively understandable.

Copula families and parameters

Edge	Copula	Parameter1	Parameter2*1	Kendall's τ	λ_U*2	λ_L*2
4-1	T	0.10	3.83	0.07	0.11	0.11
3-4	T	0.75	6.85	0.50	0.28	0.28
2-3	T	0.80	5.13	0.59	0.44	0.44
9-11	Frank	4.47	-	0.42	-	-
9-2	Frank	8.74	-	0.63	-	-
5-13	Frank	1.68	-	0.18	-	-
8-6	Gumbel	1.59	-	0.37	0.45	-
8-5	Gaussian	0.65	-	0.45	-	-
7-8	Frank	4.98	-	0.46	-	-
9-7	T	0.64	30.00	0.44	0.01	0.01
9-10	Serval Gumbel	2.32	-	0.57	-	0.65
12-9	Frank	8.13	-	0.61	-	-

*1 The parameter2 of t copula is a degree of freedom.

*2 λ_U and λ_L denote upper and lower tail-dependence coefficient respectively.

The different copula is applied to each pair of variables.

The vine copula model can capture complex dependence, especially the tail dependence.

Comparison with benchmark copulas

- The performance of the vine copula is the best in all criteria.
- The Frank Copula and the Clayton Copula have only one parameter, and cannot express the difference by the pairs of variables.
- The Gaussian copula and the t copula can set different correlation coefficients for each pair of variables, but the dependence is symmetrical and linear.

	Log-likelihood*1	AIC*2	BIC*2
Vine copula	415.4	-702.9	-540.8
Gaussian copula	313.3	-476.4	-273.1
T copula	342.4	-526.8	-326.7
Frank copula	112.2	-222.5	-219.9
Clayton copula	68.6	-135.1	-132.6

*1 Larger is better.

*2 Smaller is better.

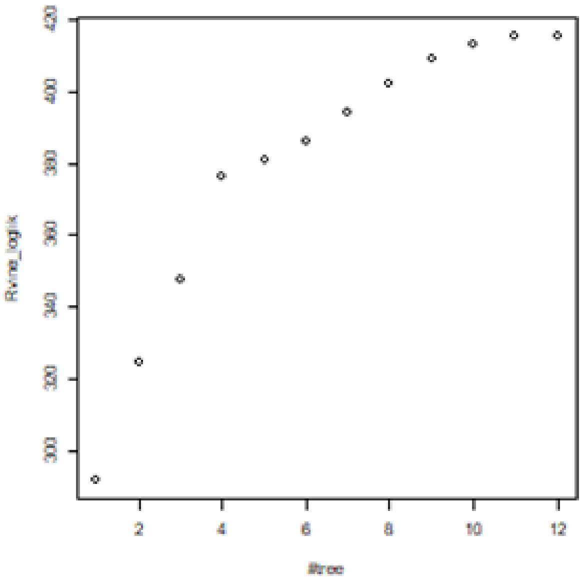
Analysis of the contribution by copulas in tree 2 and above

In order to investigate how the copulas associated with the higher trees contribute,

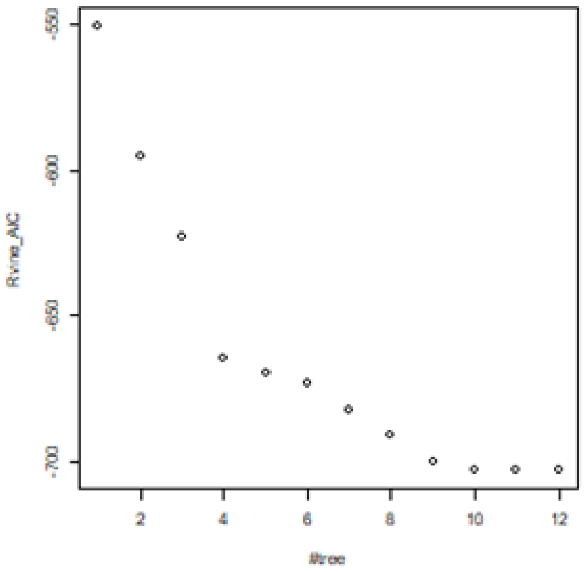
We set vine copula models consisted of the low-dimensional trees and plot the log-likelihood, AIC, and BIC.

BIC is the best when we assume independence for the tree 5 and above.

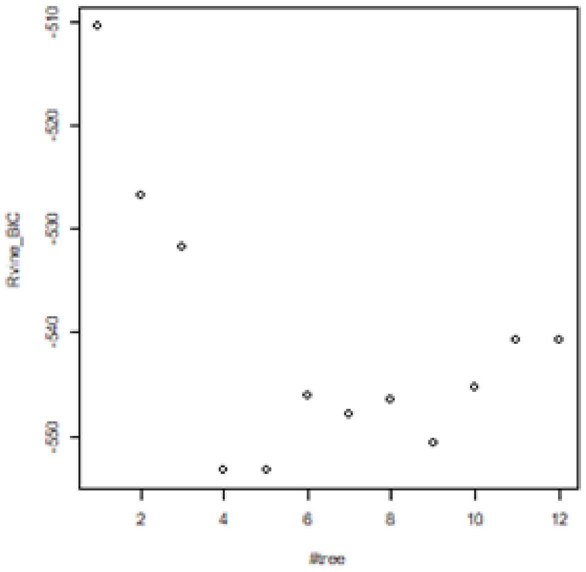
Log-likelihood



AIC



BIC



Simulation

Simulation

- We defined a survival index by the equation below,

$$ {}_{90}S_{65} = \sum_{j=1}^{n} w_j \prod_{x=65}^{90} \left(1 - q_x^j\right) $$

- Following the capital regulations, such as ICS and Solvency Ⅱ, we evaluate 99.5% Value at Risk (VaR) over the one year time horizon.
- Estimate the movement of ${}_{90}S_{65}$ after 1 Yr (2,000 times) and evaluate the difference by copula models(vine, Gaussian, and t) in the stress situation.
- We conduct the simulation of the following two cases.

Case1: All exposures are evenly diversified in all countries.

Case2: All exposures are concentrated in Group 1 (Belgium(BEL), Netherlands (NLD), and England & Wales(EW)).

Case 1 All exposures are evenly diversified in all countries

- No significant difference in the 95%ile and 5%ile.(red square)
- The 99.5%ile value of the t copula is the largest, the vine copula is middle, and the Gaussian copula is the smallest.(blue square)
- The Gaussian copula is minimum due to its no tail dependence.
- The vine copula consists of a pair of copulas with and without tail dependence, so the tail risk is larger than a Gaussian copula.
- The t copula shows the highest tail dependence because the dependence of all countries increases in the tail environment.

t copula > vine copula> Gaussian copula

No significant difference

	99.5%ile	95%ile	50%ile	5%ile	0.5%ile
Vine copula	1.114	1.060	1.000	0.941	0.890
Gaussian copula	1.111	1.060	1.000	0.944	0.889
T copula	1.122	1.058	1.000	0.944	0.893

Case 2 All exposures are concentrated in Group 1

- Similar to case 1, no significant difference in the 95%ile and 5%ile.(red square)
- However, different from case 1, the 99.5%ile value estimated by the vine copula is the largest. (blue square)
- All pair of the countries in Group 1 are modeled by the t copula with stronger tail dependence than the t copula model.
- We underestimate the dependence of the tail of concentrated exposures by Gaussian copula or t copula.

vine copula > t copula > Gaussian copula

No significant difference

	99.5%ile	95%ile	50%ile	5%ile	0.5%ile
Vine copula	1.233	1.102	1.000	0.896	0.808
Gaussian copula	1.193	1.105	1.000	0.900	0.812
T copula	1.219	1.103	1.000	0.900	0.806

Summary

- We demonstrate the usefulness of a vine copula using actual data by following steps,
  - Use a Lee-Carter model to estimate the mortality rates of 13 countries (12 European countries and Australia).
  - Model the dependence among the time-varying mortality improvement parameters of each country by the vine copula.
  - Demonstrate that the vine copula is superior on some measures (Log-likelihood, AIC, and BIC) to other benchmark copulas.
- We obtain a dependence structure resembling the actual geographical relationships, which is intuitively understandable.
- This study reveals that we underestimate the dependence of the tail risk of concentrated exposures in multiple countries by Gaussian copula or t copula.

Thank you for your attention

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Appendix

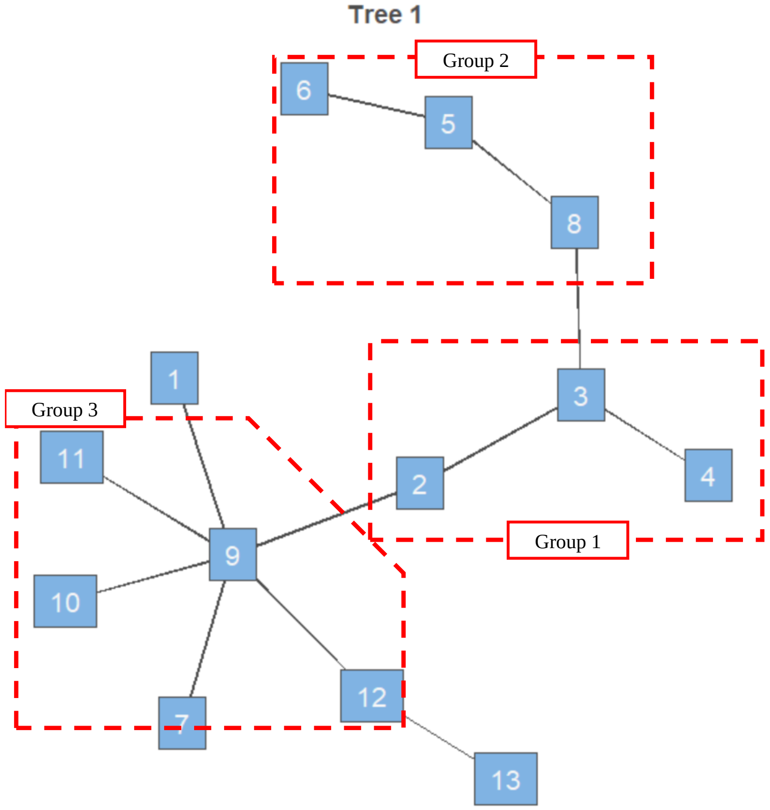
Verification of robustness by applying to other data

To examine the robustness, We apply same estimation procedure for the following three patterns.

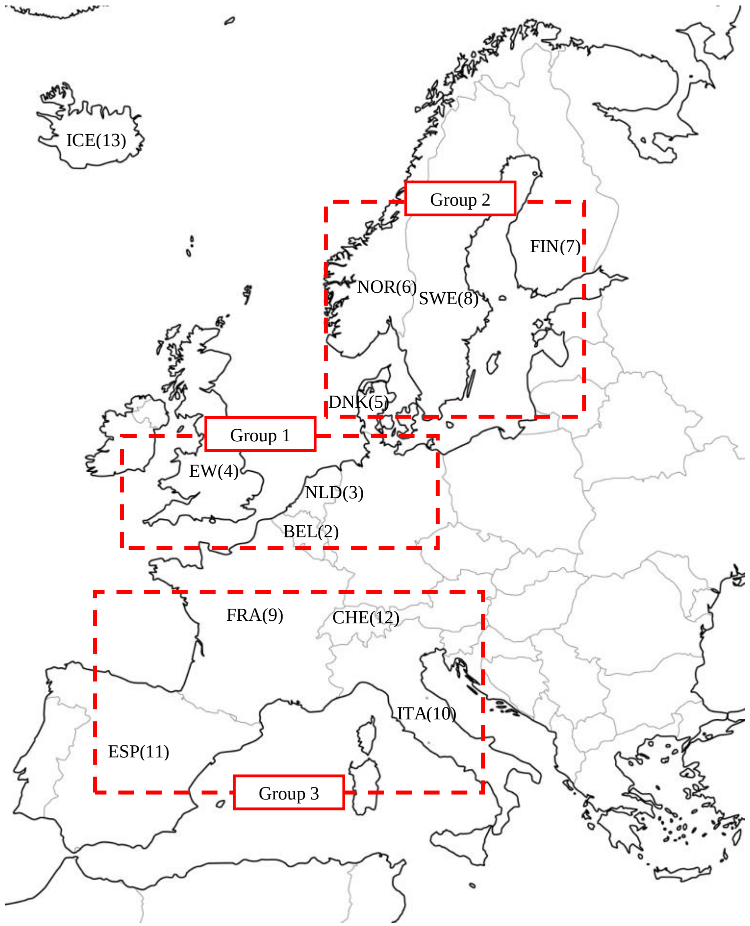
- ➢Case 1 uses female data of the same age and period
- ➢Case 2 uses the post-war period (1955-2014)
- ➢Case 3 uses working-age population (15-64)

	Sex	Year	Age
Case 1	Female	1921-2014	65-90
Case 2	Male	1955-2014	65-90
Case 3	Male	1921-2014	15-64

Case 1: Using female data of the same age and period

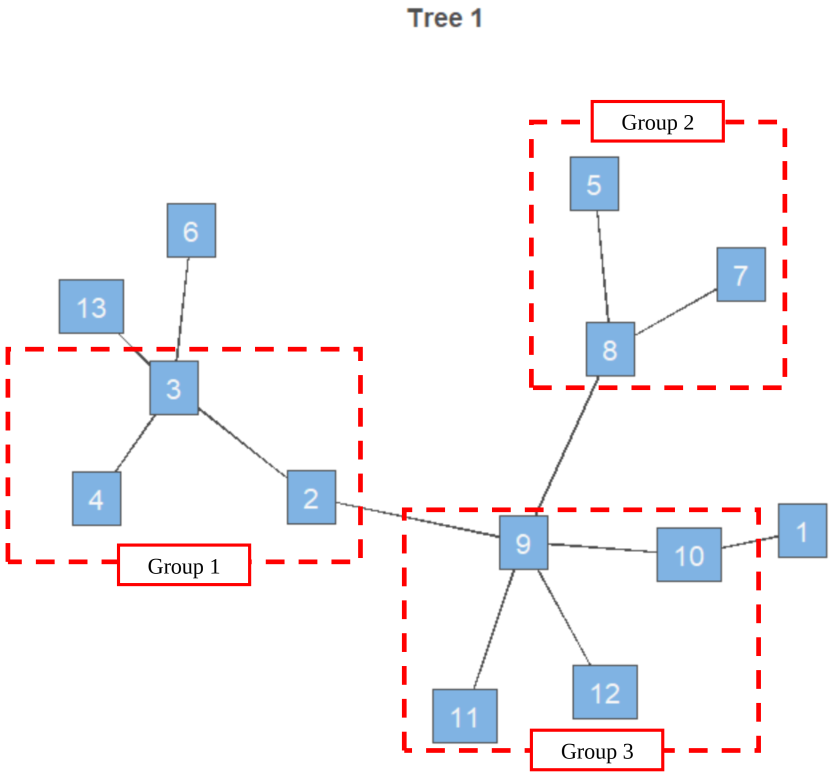


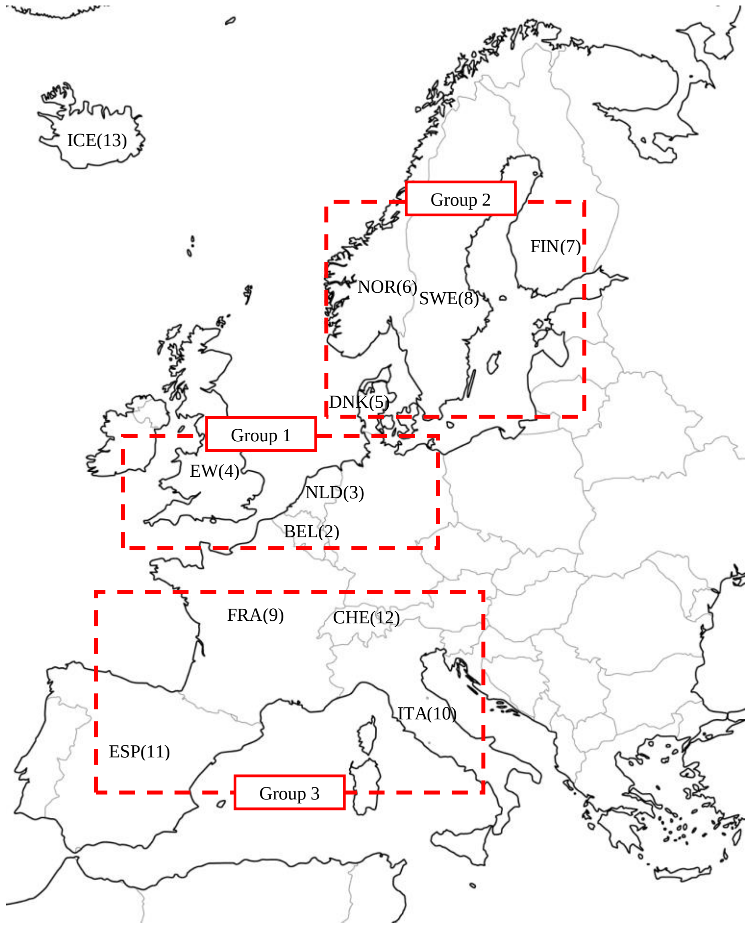
1 <-> AUS, 2 < -> BEL, 3 < -> NLD, 4 <-> EW, 5 < -> DNK, 6 < -> NOR, 7 < -> FIN,
8 <-> SWE, 9 < -> FRA, 10 < -> ITA, 11 < -> ESP, 12 < -> CHE, 13 < -> ICE



Aside form Finland(7), the dependence structure resembles the actual geographical location.

Case 2: Using the post-war period (1955-2014)

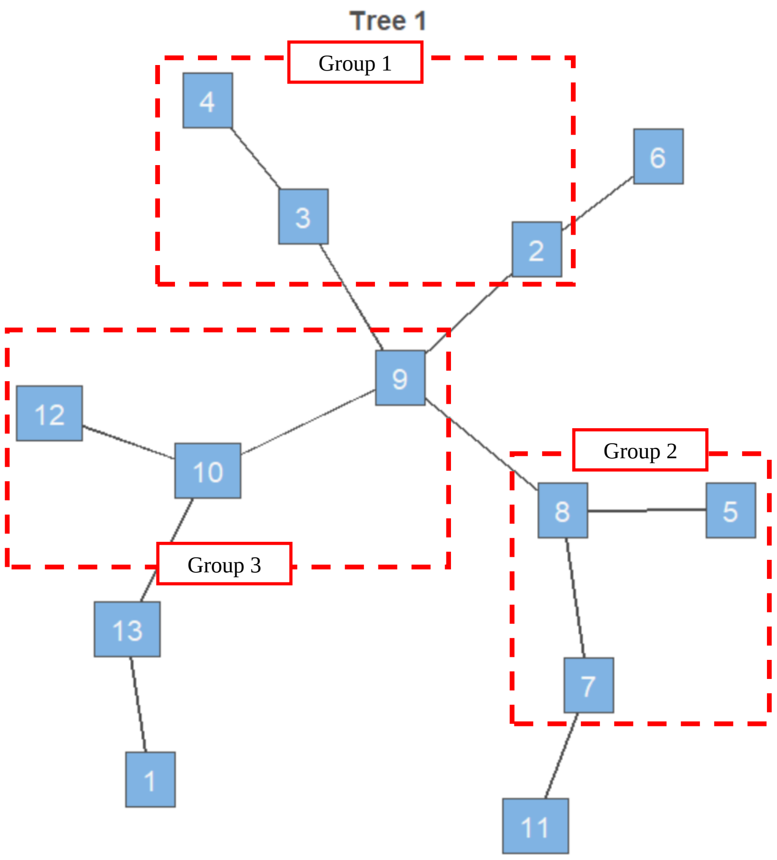


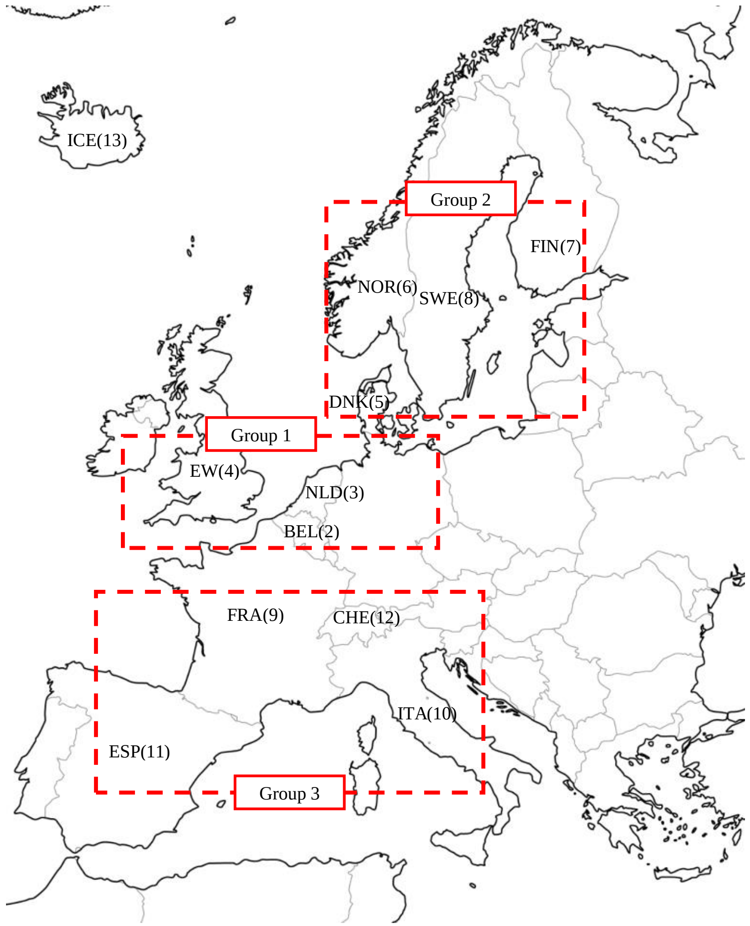


1 <-> AUS, 2 <-> BEL, 3 <-> NLD, 4 <-> EW, 5 <-> DNK, 6 <-> NOR, 7 <-> FIN,
8 <-> SWE, 9 <-> FRA, 10 <-> ITA, 11 <-> ESP, 12 <-> CHE, 13 <-> ICE

Aside form Norway(6), the dependence structure resembles the actual geographical location.

Case 3: Using working-age population (15-64)





1 <-> AUS, 2 < -> BEL, 3 < -> NLD, 4 < -> EW, 5 < -> DNK, 6 < -> NOR, 7 < -> FIN,

8 <-> SWE, 9 < -> FRA, 10 < -> ITA, 11 < -> ESP, 12 < -> CHE, 13 < -> ICE

Aside form Norway (6) and Spain (11), the dependence structure resembles the actual geographical location.